Statistics for Continuity and Differentiability: An Application to Attractor Reconstruction from Time Series

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Abstract.

Given a sampling of pairs of points from a function's domain and range what can we say about any mathematical properties, if any, of that function? For example, were the point pairs related by a function that is continuous? Or differentiable? This is the same situation we are faced with when we apply standard statistical tools, like linear correlation, except here we are asking more fundamental questions about the properties of a functional relation, rather than the form (linear, polynomial, etc.). We show here several statistics that provide likelihood or confidence measures of each functional property given samplings of point pairs. We apply these statistics to time-series reconstructions of attractors. An important observation is that many commonly asked questions about properties of such reconstructions are actually questions about more fundamental mathematical properties. Such questions include, determinism in a time series, general synchronization between attractors, the relation of a filtered or transformed attractor to the original one, and whether two time series come from the same dynamical system (dynamical interdependence).

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I. Introduction

Statisticians are often faced with the problem of having pairs of points sampled from a process or experiment. The immediate question often is, what is the relationship, if any, between the point pairs? This is often viewed as the question, what kind of function relates the two data sets? A typical test is a calculation of the linear correlation. If this quantity is near 1.0, the data sets are considered linearly related. This is generalizable to multivariate data, say $\mathbf{x} \quad \mathrm{R}^n$ and $\mathbf{y} \quad \mathrm{R}^m$, then $\mathbf{x} = \mathbf{A}\mathbf{y}$ is the possible functional relationship to be tested. Other statistical quantities and tests have been developed for linear correlation. However, low linear correlation does not imply other relationships do not exist.

We propose a more general test than testing for one type of functional model. We show that it is possible to start with mathematical definitions of function properties and construct a statistic which can be calculated from a sampling of point pairs from the function's domain and range. The statistic would then indicate whether the points are related by a function with that property.

For example, we construct a statistic to test for the existence of a continuous function. Note, we assume no particular functional form. We adhere strictly to the definition of continuity in constructing our statistic. A simple example of a situation where this test would be applied is shown in Fig. 1. There we have a sampling of points from the domain (x) and range (y) of a possible function. We ask the question what is the likelihood that these points were sampled from a continuous function f: $x \rightarrow y$?

We can also ask the same question about differentiability, namely, were these points sampled from a differentiable function? Again, we create a statistic by adhering to the mathematical definition of differentiability.

We can generalize this approach to functions on higher dimensional manifolds. One place this

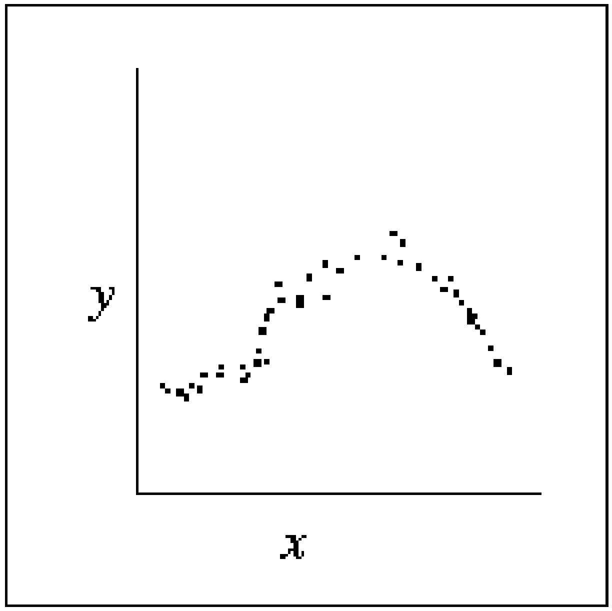

Fig. 1 A sampling of points from a function

becomes important (and the original motivation for this work) is in the analysis of time series reconstruction of attractors. Attractors exist on manifolds and when we have two simultaneously measured time series we have the potential of their being related by a function. One attractor can be considered in the domain of the function (we will refer to that as the X space) and the other in the range of the function (the Y space).

The importance of testing for continuity and differentiability come from Taken's theorem (11, 21, 22) and also from rephrasing common questions in mathematical terms. The former is a statement that one can reconstruct the trajectory of a dynamical system in phase space from a single time-series of data from the system. In fact the theorem states that there is a diffeomorphism between the reconstructed trajectory and the system's actual trajectory in its physical phase space

– a diffeomorphism is a continuous, differentiable mapping between geometric objects whose inverse is continuous and differentiable, too. Thus, for example, if we have two simultaneously measured time-series in an experiment we might want to see if there is a diffeomorphism between them. If so, they are measurements of the same dynamical system, by Taken's theorem. In this way we would have answered the "colloquial" question, are the time series from dynamically interacting parts of the experiment?

Other colloquial questions would include:

- Given one time series, does it come from a deterministic system? Such a system has the characteristic that points nearby in phase space stay nearby at later times (predictability). This is just a restatement of continuity in forward time. Other determinism questions (smoothness?) can also be raised. (For earlier related work on this see Ref. (7).)
- Does a transformation of a time series (e.g. filtering) preserve certain dynamical invariants? An example of this is fractal dimension which is preserved under a C^1 transformation (20).
- Are two parts of a dynamical system in general synchronization? This would imply a one-to-one (smooth, perhaps) transformation would exist between their trajectory reconstructions (1, 17). This is the same as asking if the trajectories are related by a continuous (and differentiable) and inversely continuous (and inversely differentiable) function.

In statistics we may ask the "colloquial" question, is a measurable variable *y* predictable from values of other (multivariate) variables, say **x**? We usually mean here that when the multivariate variables **x** are near a set of values they all had before, then *y* is near the value it had for the prior cases, too. This is shown schematically in Fig. 2 and is the *minimum* we would hope for in order to have any theory or data fitting.

We note that although others have suggested statistical quantities which show relations between data sets and reconstructed attractors (8, 16, 17), with the exception of Kaplan (7), none attempts to determine the actual mathematical properties of the function (if any) relating the data sets as we do here.

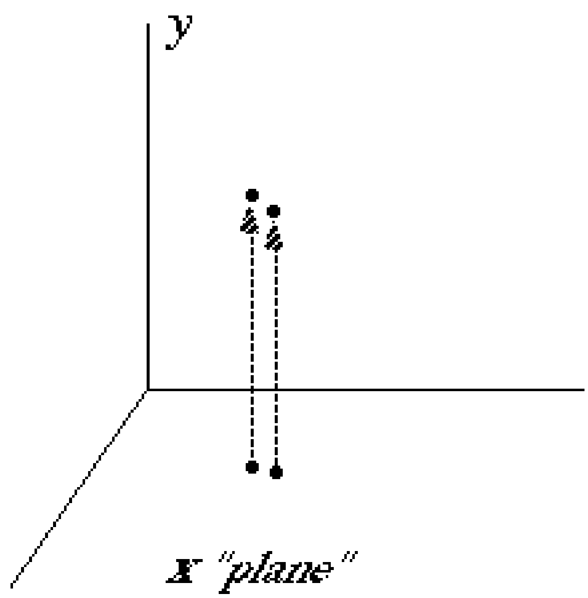

Fig. 2 Schematic diagram showing predictable dependence of a variable *y* on other multivariate data **x**, the latter being represented by the flat plane. Nearby **x** points map to nearby *y* values.

In the next section we derive the statistics for testing for continuity and differentiability. We refer the reader to our longer paper for more details (12).

II. Topological Statistics

A. Continuity statistic

Our goal is to develop simple statistics that (1) are clearly connected to the one mathematical property they measure and (2) have a separate statistical hypothesis associated with them. Criterion number (1) guarantees that we are sure what property we are testing. Number (2) guarantees we know what we are using as a gauge of that property. These goals are, in a sense, reversed from what one usually finds in the literature where tests which are an amalgam of mathematical properties and numerical algorithms are used to test for attributes which are derived. Although these numerical tests can be useful, it is not always clear what mathematical property is being tested.

We are testing for continuity (C^0) and differentiability (C^1). We give two separate statistical tests, one for each property.

A test for continuity must start with the definition of continuity: the function **f** is continuous at a point $\mathbf{x}_0$ if $\quad \varepsilon>0 \quad \delta>0$ such that $\| \mathbf{x}-\mathbf{x}_0\|<\delta \quad \|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)\|<\varepsilon$. In simpler terms if we restrict ourselves to some local region around $\mathbf{f}(\mathbf{x}_0)$, then there must exist a local region around $\mathbf{x}_0$ all of whose points are mapped into the $\mathbf{f}(\mathbf{x}_0)$ region. From this definition we can generate an algorithm to apply to time-series reconstructions.

Our reconstructions are matched in a "one-to-one" fashion (which we call **f**) in that for every point **x** we have a corresponding point **y**; however, we do not know how points in local neighborhoods are paired. We choose an ε-sized set around a fiducial point $\mathbf{y}_0$, we also choose a δ-sized set around it's preimage $\mathbf{x}_0$. We check whether all the points in the δ set map into the ε set. If not, we reduce δ and try again. We continue until we run out of points or all points from a small-enough δ set fall in the ε set. We count the number of points in the ε set (n_ε) and the δ set (n_δ). We do not include the fiducial points $\mathbf{y}_0$ or $\mathbf{x}_0$, since they are present by construction. Generally

$n_\varepsilon \geq n_\delta$.

We now choose the null hypothesis which helps us generate a probability that one should find n_ε and n_δ points in such an arrangement. Many null hypotheses are possible. We choose the simplest, namely, that placements of the points on the **x** and **y** attractors are independent of each other. This null hypothesis is not trivial. It is typical of what one would like to disprove early on in any attractor analysis, namely that the data have a relation to each other. We will see that it can show much more.

Given the null hypothesis we approximate the probability of a point from the δ set falling at random in the ε set as $p = n_\varepsilon / N$, where N is the total number of points on the attractor. Then the probability that n_δ points will fall in the ε set is p^{n_δ}. We obtain a likelihood that this will happen by taking the ratio of this probability to the probability for the most likely event, p_{binmax}. The latter is just the maximum of the binomial distribution for n_δ points given probability p for each individual event. We see that p^{n_δ} is simply the "tail-end" of the binomial distribution. The maximum generally will occur for some intermediate number of δ points, say m ($< n_\delta$), falling in the ε set. If $p^{n_\delta} \ll p_{\text{binmax}}$, then the null hypothesis is not likely and can be rejected.

We define the continuity statistic as $\Theta_{C^0} = 1 - p^{n_\delta} / p_{\text{binmax}}$. When $\Theta_{C^0} \approx 1$ we can confidently reject the null hypothesis. The points in the ε set are behaving as though they are generated by a continuous function on the δ set. When $\Theta_{C^0} \approx 0$ we cannot reject the null hypothesis and the points are behaving as though they are independent. Note that if we run out of points ($n_\delta = 0$), then we usually take the logical position that we cannot reject the null hypothesis and set Θ_{C^0}

$= 0$. $\Theta_C 0$ will depend on ε, the resolution, and we will examine the statistic for a range of ε's. To get a global estimate of the continuity of **f** on the attractor we average $\Theta_C 0$ over the entire attractor or over a random sampling of points on it. We present those averages here.

For inverse continuity (i.e. continuity of $\mathbf{f}^{-1}$) we simply reverse the roles of the spaces (X and Y). Now we examine the mapping of δ set points in Y into ε sets in X. The statistical arguments are the same. We generate an inverse continuity statistic $\Theta_I 0$ which has the same meaning as $\Theta_C 0$, except that it applies to $\mathbf{f}^{-1}$: $Y \rightarrow X$.

B. Differentiability Statistic

The differentiability statistic is generated in the same vein as the continuity statistic. We start with the mathematical definition of a derivative and apply it locally to the two reconstructions. The generation of the linear map that approximates the derivative and the likelihood estimate associated with it are more complex than for continuity. We show the details in our longer paper (12). We only outline the differentiability statistic here.

The definition of a derivative at a point $\mathbf{x}_0$ is that a linear operator $\mathbf{A}$ exists such that $\varepsilon > 0$ $\delta > 0$ for which $\| \mathbf{x} - \mathbf{x}_0 \| < \delta$ $\| \mathbf{f}(\mathbf{x}_0) + \mathbf{A}(\mathbf{x} - \mathbf{x}_0) - \mathbf{f}(\mathbf{x}) \| < \varepsilon \| \mathbf{x} - \mathbf{x}_0 \|$. This means that there is a linear map that approximates the function at nearby points with an error in the approximation that is proportional to the distance between those points. Note that ε serves a purpose here different from continuity. Here it bounds the error in a linear mapping; it does not signify a distance.

The algorithm that we generate from this definition is to first choose an ε (error bound) and a δ. Then we find all the points in the local δ set $\{\mathbf{x}_i\}$ and their **y** counterparts $\{\mathbf{y}_i\}$. We approximate the linear operator **A** as the least squares solution of the linear equations $\mathbf{A}(\mathbf{x}_i - \mathbf{x}_0) = (\mathbf{y}_i - \mathbf{y}_0)$. The solution is accomplished by singular value decomposition (SVD) (12). The least-squares approach appears to be the best since it minimizes the errors in the approximation which is just what ε is supposed to bound. We check if $\| \mathbf{y}_i - \mathbf{y}_0 - \mathbf{A}(\mathbf{x}_i - \mathbf{x}_0) \| < \varepsilon \; \| \mathbf{x} - \mathbf{x}_0 \|$ for each i. If not, we decrease δ and try again with fewer, but nearer points. We continue this until we have success or we run out of points.

We choose the null hypothesis that the two sets of vectors $\{\mathbf{x}_i\}$ and $\{\mathbf{y}_i\}$ have zero correlation. We show (12) that this generates a likelihood that any two such sets will give the operator $\mathbf{A}$ "by accident" as $e^{-\frac{1}{2}(n_\delta - r_x)(n_\delta - r_y) r^2 d}$, where r^2 is the usual multivariate statistical correlation between $\{\mathbf{x}_i\}$ and $\{\mathbf{y}_i\}$, $\delta = \min(r_x, r_y)$, and r_x and r_y are the ranks of the $\mathbf{x}$ and $\mathbf{y}$ spaces that come out of the SVD (12). This is an asymptotic formula. The differentiability statistic $\Theta_{C}1$ is given by one minus this likelihood. When $\Theta_{C}1 \approx 1$ we can reject the possibility that the points are accidentally related by a linear operator, a derivative. When $\Theta_{C}1 \approx 0$, we cannot reject the null hypothesis. As before, when we shrink δ so small that no points other than $\mathbf{x}_0$ remain, we set $\Theta_{C}1 = 0$. Analogous to $\Theta_{C}0$, the statistic $\Theta_{C}1$ depends on ε. We typically calculate $\Theta_{C}1$ for a range of ε's and average over the attractor or over a random sampling of points on it.

We now have two statistics $\Theta_{C}0$ and $\Theta_{C}1$ that we can use to test for continuity and differentiability in data sets. We will mostly concentrate on the continuity statistic, since the differentiability statistic follows it very closely in many cases, although we show one example for which this is not the case. We give all ε and δ values in units of the standard deviation of the reconstructed attractor, so that, for example $\varepsilon = 1.0$ is an ε set with radius of one standard deviation. For all of our studies so far this seems to be a good normalization for set sizes.

Similar to $\Theta_{I}0$ we develop a statistic for the differentiability of the inverse function $\mathbf{f}^{-1}: Y \rightarrow X$, by reversing the roles of X and Y. We now examine linear maps on δ sets in Y which map to nearby points in X. The inverse differentiability statistic is called $\Theta_I 1$ and it has the same meaning as $\Theta_C 1$.

C. Putting it all together

We are now in a position to make statistical arguments for pairs of (multivariate) data points being related by a function which may have several different properties. It may be simply continuous. It may have a continuous inverse. It may have both continuities, in which case we can make an argument that it is a homeomorphism and all topological quantities will be invariant under it, i.e. whatever topological property the X data points have, the Y data points have.

Similarly, should the function pass continuity tests, we can test for differentiability of the function or its inverse. Should both differentiability statistics be high (≈1), then we can argue that we have data points related by a diffeomorphism, i.e. local metric properties (like fractal dimension) are preserved.

We should always keep in mind that we are not making statements about the absolute continuity or differentiability of function, but only the continuity or differentiability relative to the smallest ε's and δ's for which are statistics are good. That is, we always have a resolution limit associated with an argument for these properties. A good question is, if a function can be shown to be continuous or differentiable at some level of resolution, are all topological properties that can be calculated at or above that resolution invariant? We suspect the answer is 'yes,' but we cannot prove it.

III. Applications

A. Continuity vs. linear correlation Lorenz x and z time series

To help clarify the use of these statistics we first present two cases, one in which we compare

$\Theta_C 0$ to the usual statistical linear correlation and another in which we compare $\Theta_C 0$ to $\Theta_C 1$.

Fig. 3 shows a plot of *x* and *z* time series taken from a Lorenz system running in the chaotic regime (σ=10, *b*=8/3, and ρ=60). If we calculate the linear correlation or cross correlation between these time series for various time delays (including zero) we will find that the correlation is very small. For example for 5000 points it is less than 0.085 for any time lag. The multivariate linear correlation calculated from time-delay reconstructions using the *x* and *z* series would show the same low correlation, as is easy to see from the form of the time-delay method.

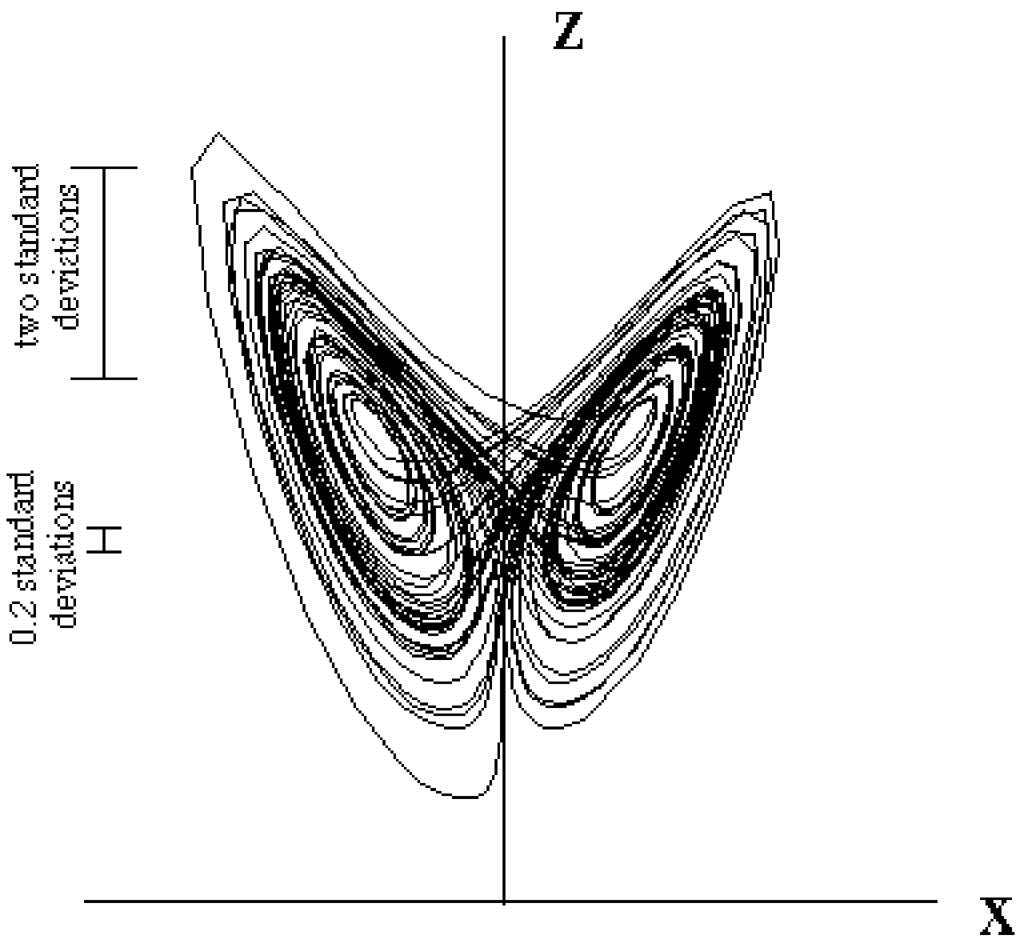

Fig. 3 Plot of the *x* and *z* time series, along with two measure bars showing the size of the maxi-

mum ε set used and the smallest ε set at which the continuity statistic from *x* to *z* was still near 1.0.

If we now calculate $\Theta_C 0$ using 4-dimensional reconstructions from the *x* and *z* series (5000 points) we see something entirely different. Fig. 4 shows $\Theta_C 0(\varepsilon)$ vs. ε. The statistic is quite high (> 0.8) down to an ε of about 0.1 standard deviation. Fig. 3 shows the size of ε sets of two standard deviations in diameter and of 0.2 standard deviations in diameter. This shows that down to a fairly small resolution, with a small number of data points, we can establish that the mapping between the *x* attractor and the *z* reconstructed attractor is continuous, i.e. **f**: *x* attractor $\rightarrow$ *z* attractor is continuous. It turns out that the differentiability statistic $\Theta_C 1$ is also quite high down to errors in the linear operator on the order of 0.2.

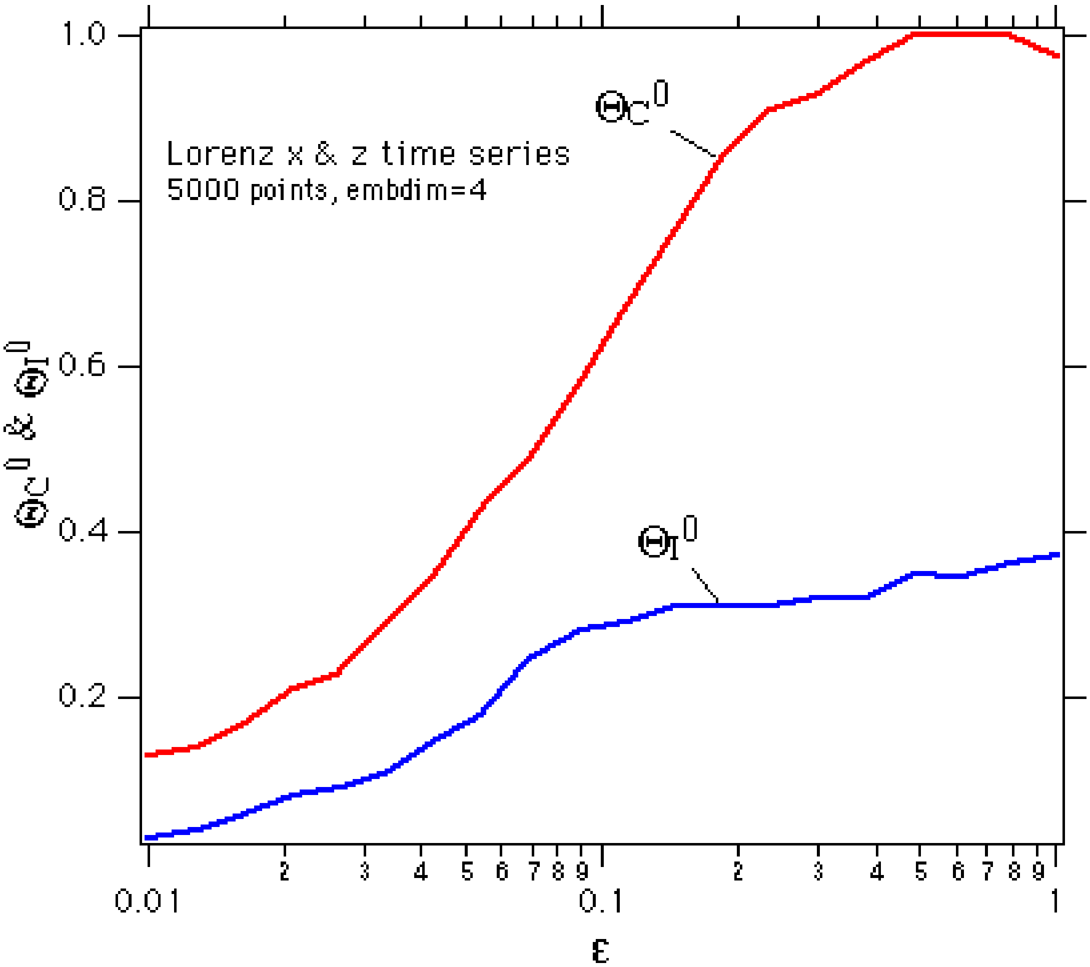

Fig. 4 Continuity and inverse-continuity statistics for the x attractor to z attractor mapping.

On the other hand, the inverse map $\mathbf{f}^{-1}$: z attractor $\rightarrow$ x attractor is not continuous (see Fig. 4). This is not surprising since the z component of the Lorenz system has a two-to-one relation to the x and y components in that the vector field is invariant to changing the signs of x and y. This means that $\pm x$ values will map to roughly the same point on the z attractor or, equivalently, each point on the z attractor maps back to two points on the x attractor. This is a discontinuity and the statistic shows this. The statistic $\Theta_I 0$ is not zero since sets of finite δ size near the x origin will have points that *can* map into the same ε set.

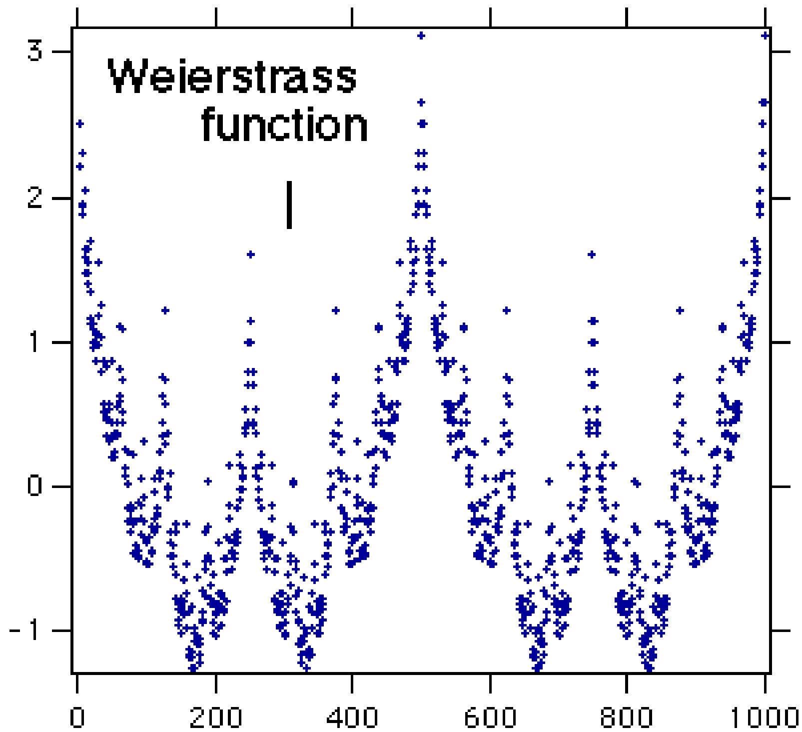

Fig. 5 A plot of the Weierstrass function used to test the continuity and differentiability statistics.

B. Differentiability of sampling of Weierstrass function

In our next test we examine a function that is continuous, but not differentiable, the Weierstrass function. We show this function in Fig. 5. The “time series” are just the points on the abscissa (X) and the corresponding function values on the ordinate (Y). Here there are no attractors to reconstruct from the time series; instead there is only a function from $\mathbf{R} \rightarrow \mathbf{R}$. This shows that the statistics can be applied to any function given pairs of points in the domain and range.

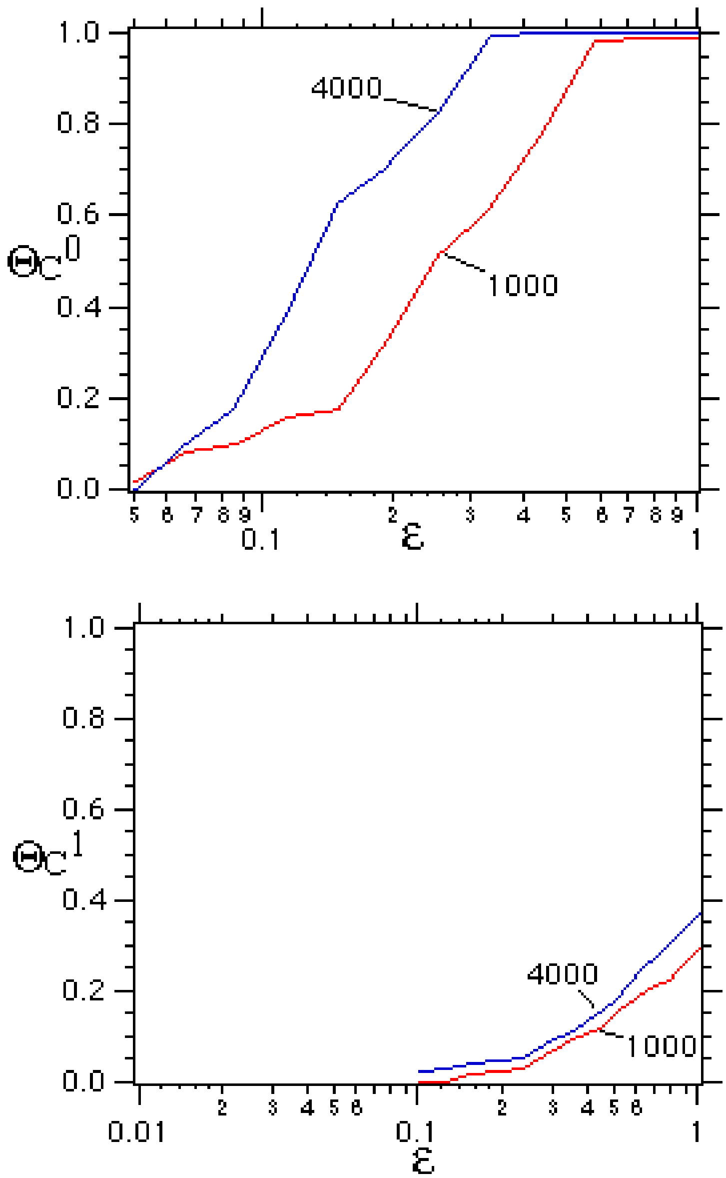

Fig. 6 Continuity and differentiability statistics for the Weierstrass function.

The continuity and differentiability statistics are shown in Fig. 6. The continuity statistic even for only 1000 points starts high and degrades slowly with decreasing ε. As we add more points (up to 4000, in this case) the continuity improves. This is also a characteristic of a functional relation that is truly continuous. A function that has many discontinuities will not have Θ_C0 increase as the amount of data increases.

On the other hand, the differentiability statistic starts very low and quickly goes to zero. This indicates that the function is not differentiable (we cannot reject the null hypothesis for $\Theta_C 1$). Thus, $\Theta_C 0$ and $\Theta_C 1$ do measure different properties.

C. Dynamical Interdependence

An interesting situation, especially regarding chaotic systems, is that of generalized synchronization (1). This occurs when two parts of a coupled dynamical system have the same dimension (same number of dynamical variables) and their respective trajectories or phase space points are related in a one-to-one smooth fashion. That is, there is a diffeomorphism between them.

In the usual synchronization (1-6, 13, 15, 18, 19) the subsystems of a dynamical system are equal at each time step (e.g. $x=y$ in time series). In this case the function relating the parts of the dynamical system acts like the identity mapping, which is a diffeomorphism. However, Afraimovich showed that this can be relaxed to a situation as in the previous paragraph. Recent work by Rul'kov *et al* (17) has shown that one can devise a statistic (based on the false-nearest-neighbor method (9, 10)) which can detect such general synchronization. We have shown that our continuity and differentiability statistics can do the same and at the same time specify the type of relationship (homeomorphism or diffeomorphism, for example) and the resolution at which one can be confident that the function has these properties (12).

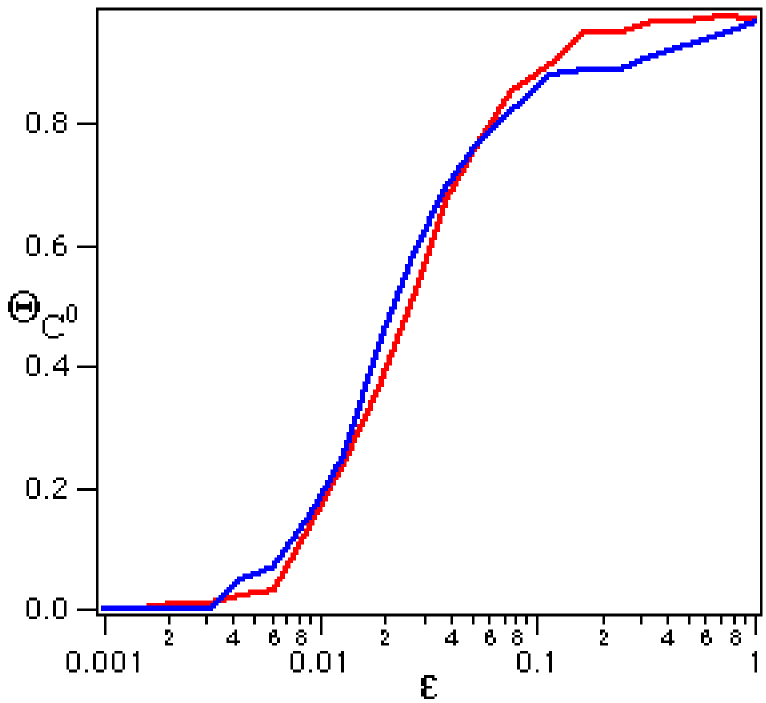

Fig. 7 Continuity statistics for the mapping between attractors reconstructed from a Lorenz *x* and a simultaneously measured Lorenz *y* time series which, combined with Fig. 8 shows that the two are related by a diffeomorphism and therefore the time series must come from the same dynamical system.

We take these ideas further here by noting that generalized synchronization, especially as determined from time series reconstructions, is really a particular type of *dynamical interdependence*. Since, generically, by Taken's theorem we can measure two different dynamical variables, say *x* and *y* time series, and reconstruct the same attractor, up to a diffeomorphism, measuring any two dynamical variables in a system should yield attractors that have high Θ_{C^0}, Θ_{C^1}, Θ_{I^0}, and Θ_{I^1}. We note this is a generic statement and there are times it will fail, as in the above test between Lorenz x and z time series.

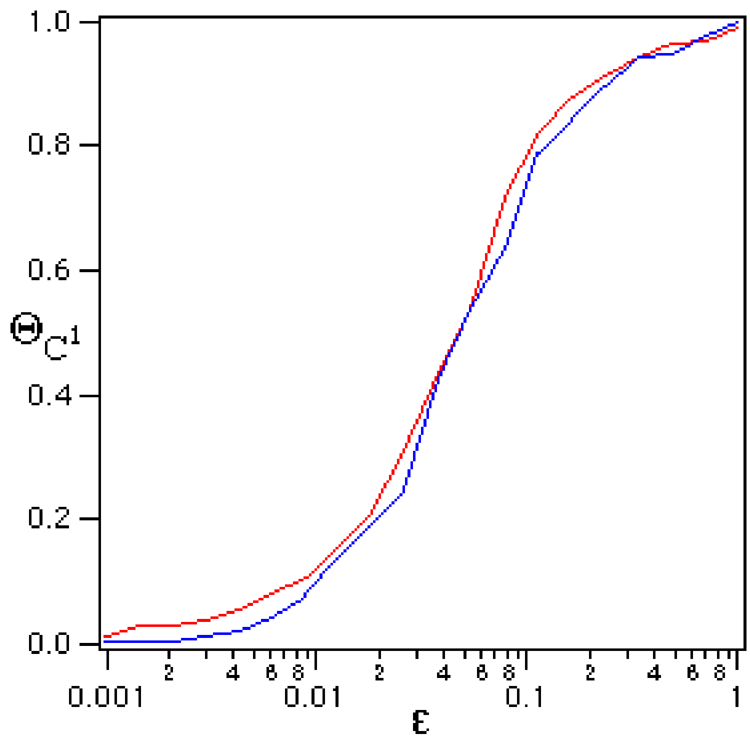

Fig. 8 Differentiability statistics for the mapping between attractors reconstructed from a Lorenz x and a Lorenz y time series which, combined with Fig. 7 shows that the two are related by a diffeomorphism and therefore the time series must come from a dynamical system.

We test the idea of dynamical interdependence on a Lorenz x and y time series (28,000 points), both of which can reproduce an equivalent Lorenz attractor. Fig. 7 shows Θ_{C^0} and Θ_{I^0} for the attractors reconstructed from the Lorenz x and y time series. We see that the continuity and inverse continuity remain high down to ε radii of ~ 0.07 standard deviations. As we saw above these are small sets on the attractor and hence we can be confident down to fairly small resolutions that our attractors are related by a homeomorphism.

Likewise the $\Theta_C 1$ and $\Theta_I 1$ statistics (Fig. 8) remain high down to errors (ε) on the order of 0.1 standard deviation. This means our function relating the two attractors is smooth and the attractors are probably related by a diffeomorphism. If these were two time series taken simultaneously in an experiment, we would conclude that they came from the same dynamical system, i.e. they are dynamically interdependent.

D. Determinism

As our last example we show how $\Theta_C 0$, $\Theta_C 1$, $\Theta_I 0$, and $\Theta_I 1$ can be used to test for determinism in a time series. In this work we are very close in spirit and sometimes in actual practice (at least as far as forward, primitive determinism) to the paper by Kaplan (7), which gives a good introduction to the general idea of using continuity properties for determinism tests.

We note here that determinism at its most primitive level simply requires phase-space points that are close in the present to be close in the future. This is forward determinism and is tested by calculating $\Theta_C 0$ for a time series reconstruction (X) and its forward, time-shifted version (Y). We can also test for forward, smooth determinism by calculating $\Theta_C 1$ on the same two time series. Invertible dynamics can be tested for by calculating $\Theta_I 0$ (primitive, reverse-time determinism) and $\Theta_I 1$ (smooth, reverse-time determinism). Thus, associated with each level of function property (where we restrict the function properties with each level), is a corresponding determinism.

Calculation of such levels of determinism could be useful if one were proposing to model a system on which experimental time series were available. This could dictate the level of model to use.

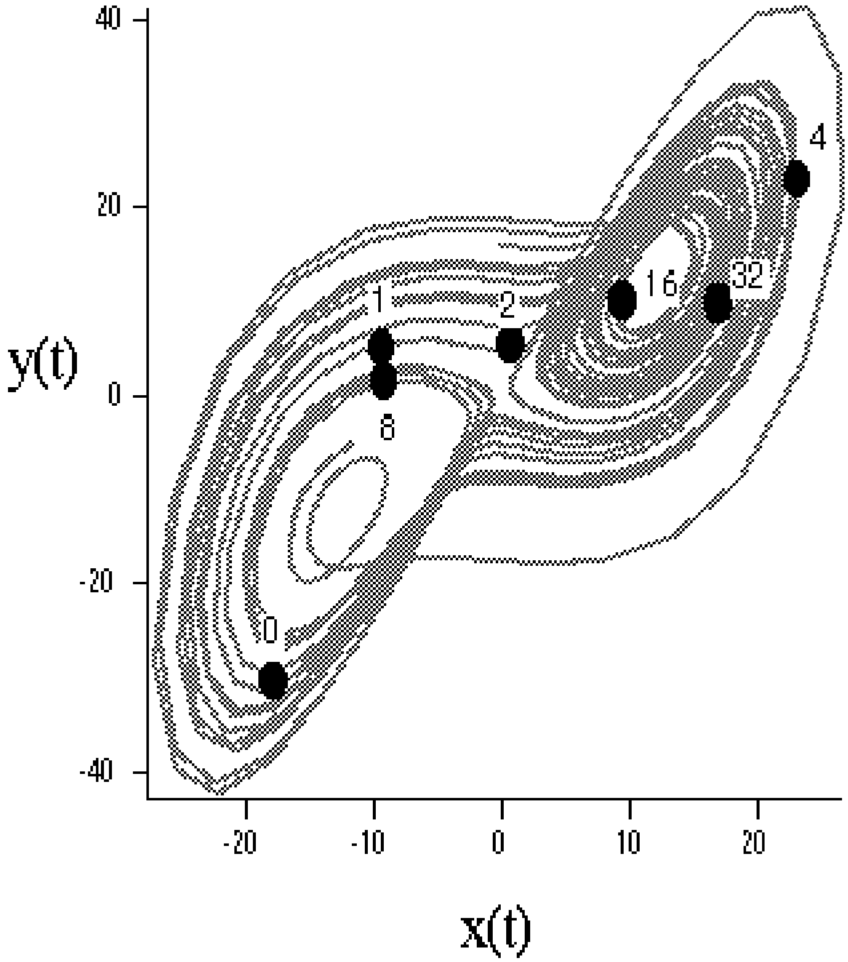

Fig. 9 The points on the Lorenz attractor in forward time where Θ_C0, Θ_C1, Θ_I0, and Θ_I1 were calculated to check for various forms of determinism.

We used a Lorenz *x* time series to test for the various levels of determinism. We chose 100 points at random on the reconstructed attractor, then calculated the average of each statistic for points both forward and backward in time at time steps of 1, 2, 4,...2^n, n=6. One time step was approximately one-quarter around the typical oscillation of the Lorenz system. Fig. 9 (from Ref. (12)) shows the points on the attractor relative to one, typical starting point. Fig. 10 (from Ref.

(12)) shows the statistics. We see that, given this time series, we would conclude that it was from a forward, smooth deterministic system. The reverse determinism is not as good, but is still present. The Lyapunov exponents determine the limits of forward and reverse determinism. The forward determinism is limited by the maximal positive exponent and the reverse determinism is limited by the minimal (in the sense of most negative) Lyapunov exponent. The most negative Lyapunov exponent of the Lorenz system is much larger in magnitude than the most positive. This probably accounts for the poorer values for the reverse determinism ($\Theta_I 0$ and $\Theta_I 1$).

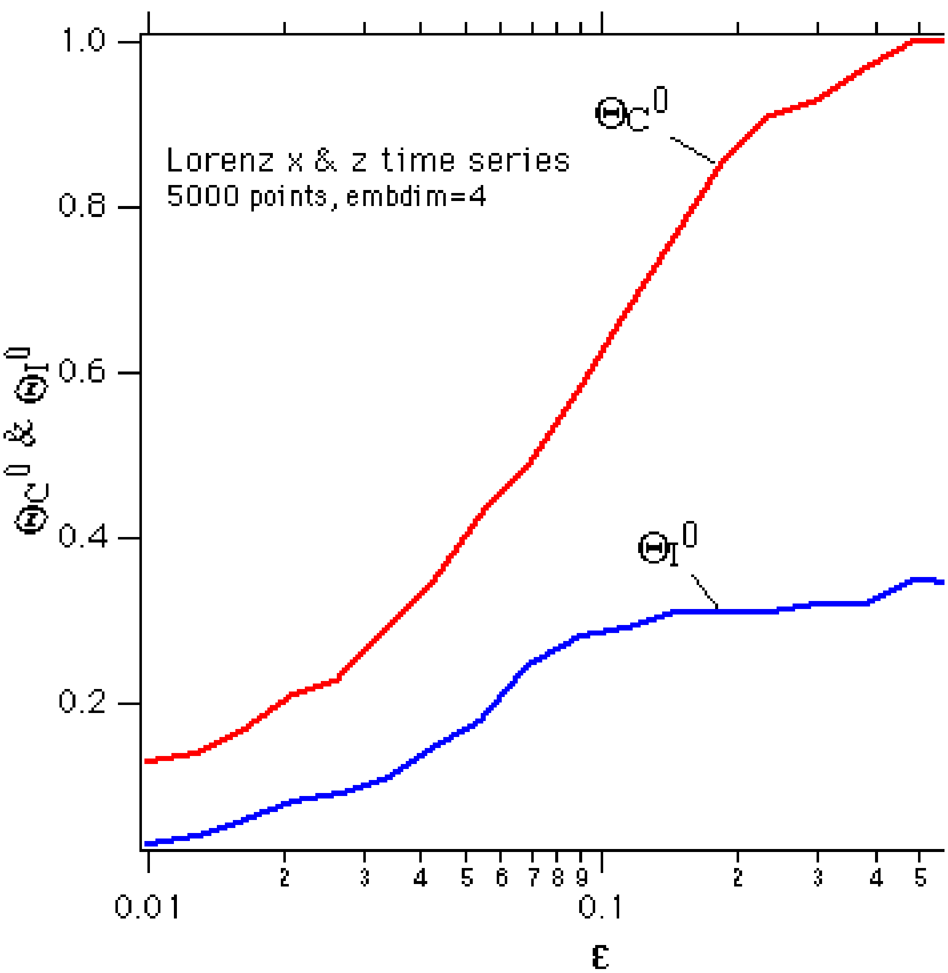

Fig. 10 The statistics $\Theta_C 0$, $\Theta_C 1$, $\Theta_I 0$, and $\Theta_I 1$ which show the level of primitive determinism,

smooth determinism in forward time and invertible primitive determinism and invertible smooth determinism in reverse time.

IV. Conclusions and Remarks

We see that we can define statistics that address the question of whether data sets are related by functions with fundamental mathematical/analytical properties. This is a more general approach than asking whether the data are related by a particular type of function (e.g. linear map) or the data are distributed according to a particular type of probability distribution (e.g. Gaussian). In testing data sets, it would seem wise to also test for general properties to determine first, whether there is even a tractable relation between data sets, and second, what level of function properties can be assigned to the relation (e.g. continuous? differentiable?).

Obviously, we can extend this work in two ways. One is to derive statistics for other mathematical properties, like twice differentiable or Lipshitz. The other way is the use other null hypotheses to generate our statistics. We have developed another continuity hypothesis, for example, which will appear elsewhere (14).

A particularly challenging direction would be to create more robust versions of the statistics and/or null hypotheses, so we can make conclusions in the presence of noise, much as is done now with linear correlations, ARMA modelling, and analyses of probability distributions. We have not investigated this possibility, but we do feel it would be very useful and in some cases absolutely necessary in order to use these statistics on actual data and experiments.

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